Estimates for the Probability Itô Processes Remain Near a Curve, and Applications

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Objects under Study

- A stochastic process: $X_t,\ 0 \le t \le T, \quad X_t \in R^n$,
- A deterministic differentiable curve: $x_t,\ 0 \le t \le T$,
- A time depending radius $R_t > 0, 0 \le t \le T$.
- $$\tau_R = \inf\{t : |X_t - x_t| \ge R_t\}$$
- $$X_t = x + \sum_{i=1}^{\infty} \int_0^{t \wedge \tau_R} \sigma_j(s, \omega, X_s) dW_s^j + \int_0^{t \wedge \tau_R} b(s, \omega, X_s) ds \qquad (1)$$

Contents

- Lower bounds for

$$P(\tau_R \geq T) = P(|X_t - x_t| \leq R_t, \text{ for all } t \leq T).$$

- Lower Bounds for expectations:

$$E(f(X_T^x)1_{B_\varepsilon(y_0)}(X_T^x))$$

- Lower Bounds for Distribution Functions:

$$P(X_T^{x,i} > y_0^i, \; i = 1, ..., n)$$

- Two main cases Elliptic and Log-Normal
- Lower Bounds for Option Pricing
  1. Call on a basket: $Y_t = \sum_{i=1}^n q_i X_t^{x,i}$.
  2. Asian Options. $Y_t = \int_0^T \sum_{i=1}^n q_i X_t^{x,i} dt$.

General Hypotheses

- Adapted: $t \to \sigma_j(t, \omega, X_t), t \to b(t, \omega, X_t)$ are adapted to the filtration of W.
- Locally Bounded:

$$|b(t \wedge \tau_R, \omega, X_{t\wedge\tau_R})| + \sum_{j=1}^{\infty} \left|\sigma_j(t \wedge \tau_R, \omega, X_{t\wedge\tau_R})\right| \leq c_t.$$

- Locally Lipshitz continuous: for every $t < s < \tau_R$

$$\sum_{j=1}^{\infty} E_t(\left|\sigma_j(s, \omega, X_s) - \sigma_j(t, \omega, X_t)\right|^2 1_{\{\tau_R < s\}}) \leq L_t^2(s - t).$$

- Locally Elliptic:

$$\lambda_t \leq \sigma\sigma^*(t, \omega, X_t) \leq \gamma_t$$

Growth Conditions

- Growth condition: The functions c_t, L_t, λ_t and γ_t belong to some $L(\mu, h)$, where
  - For $h > 0, \mu \geq 1$ we define $L(\mu, h)$ to be the class of non negative functions such that

$$f(t) \leq \mu f(s) \quad if \quad |s - t| \leq h.$$

If $f(s) = 0$ for some s then $\mu = \infty$!

Main Result

Theorem

Under the General Hypotheses

$$P(\tau_R \geq T) \geq \exp(-Q_n(1 + \int_0^T F_x(t)dt))$$

with $Q_n = 8^{4n+4}\pi^n e^2 \mu^{2n^2+20n+25}$ and

$$F_x(t) = \frac{1}{h} + c_t^2 L_t^2 \left(\frac{1}{\lambda_t} + \frac{1}{R_t}\right) + \frac{|\partial_t x_t|^2}{\lambda_t}.$$

- Osanger-Machlup function (see Ikeda-Watanabe): for an elliptic diffusion process one may compute:

$$\lim_{R\to 0} R^2 \ln P(\sup_{t\leq T} \left|X_t^x - x_t^y\right| \leq R) = c_T(x,y).$$

- Our result is not assymptotic.

Related Topics Cont.

- Lower bounds for the density of the law [2, 10, 11] (between others):

$$p_T(x, y) = P(X_t^x \in dy) \geq q_T(x, y).$$

  In fact we are not interested in the problem of existence of a density.

- Estimations of the Law of the Supremum. See, for example [7] (One dimensional case)

$$P[\sup_{s \leq T} X_s > z].$$

Step 1. Decomposition   For $t, h > 0$

$$X_{t+h} = X_t + G_{t,h} + U_{t,h}$$

$$G_{t,h} = \sum_{j=1}^{\infty} \sigma_j(t, \omega, X_t)(W^j_{t+h} - W^j_t) \qquad \textit{Gaussian}$$

$$\begin{aligned} U_{t,h} &= \sum_{j=1}^{\infty} \int_t^{t+h} (\sigma_j(s, \omega, X_s) - \sigma_j(t, \omega, X_t)) dW^j \\ &+ \int_t^{t+h} b(s, \omega, X_s) ds \quad \sim \quad h. \end{aligned}$$

Conditionally with respect to F_t, $G_{t,h}$ is Gaussian, $G_{t,h}$ is (roughly speaking) of size $\sqrt{h}$. For h sufficiently small $G_{t,h}$ is the principal term and $U_{t,h}$ is a reminder which may be *ignored.*

Problem:

How small h has to be in order to be able to ignore $U_{t,h}$.

If h tends to 0 our lower bound is of the form $P(\tau_R \geq T) \geq c^{-T/h}$. If we do so we obtain $P(\tau_R \geq T) \geq 0$ – just nothing.

Step 2. Short time behavior. For each $t \in (0, T)$, there exist $\alpha_1, \alpha_2, \alpha_3 > 0$ depending on the parameters $c_t, L_t, \lambda_t, \gamma_t$ such that, if

$$
\begin{aligned}
(i) &\quad \sqrt{h} \le \alpha_1, \\
(ii) &\quad \int_t^{t+h} |\partial x_s|^2 ds \le \alpha_2
\end{aligned}
$$

then, for $z \in R^n$ such that $|z - x_t| \le \alpha_3$ then

$$
P_t[|X_{t+h} - z| \le \eta, \tau_R > t + h] \ge K_t h^{n/2},
$$

As in Bally [2], Kohatsu [12] we work with an evolution sequence in the following sense

Lemma

- *There exists a time grid* $\{t_k\}$ *an* $h_k, \alpha_{k,1}, \alpha_{k,2}, \alpha_{k,3}$ *such that for each* k *the conditions on Step 2 are satisfied.*
- *Let*

$$A_k = \{\omega |\ |X_{t_{i-1}} - x_{t_i}| \leq \frac{1}{2}\sqrt{\lambda_{t_i} h_i}, i = 1, ..., k\} \cap \{\tau_R > t_{k-1}\}$$

Then

$$P(A_K) \geq \rho^{n/2} P(A_{k-1}),$$

- $$|b(t\wedge\tau_R,\omega,X_{t\wedge\tau_R})|+\sum_{j=1}^{\infty}\left|\sigma_j(t\wedge\tau_R,\omega,X_{t\wedge\tau_R})\right|\le c.$$
- Locally Lipschitz continuous, for every $s<t<\tau_R$

$$\sum_{j=1}^{\infty}E_t(\left|\sigma_j(t,\omega,X_t)-\sigma_j(s,\omega,X_s)\right|^2 1_{\{\tau_R<t\}})\le L^2(t-s).$$

- Locally Elliptic

$$\lambda\le\sigma\sigma^*(t,\omega,X_t)\le\gamma.$$

The bounds do not depend on time

Theorem

In the Elliptic case we have

$$P(\sup_{t\leq T}|X_t - x_t| \leq R) \geq \exp(-C_T(K + \frac{|y-x|^2}{\lambda T})).$$

In particular:

$$P(|X_T - y| \leq R) \geq \exp(-C_T(K + \frac{|y-x|^2}{\lambda T})).$$

where, we take $y \in R^n$ and the straight line

$$x_t = x + \frac{t}{T}(y - x),$$

Proof

For the straight line, we have $|\partial_t x_t| = \frac{1}{T}|y-x|$ so

$$F_x(t) = \frac{1}{h} + c^2L^2(\frac{1}{\lambda} + \frac{1}{R}) + \frac{|y-x|^2}{\lambda T^2}$$

and

$$\int_0^T F_x(t)dt = T(\frac{1}{h} + c^2L^2(\frac{1}{\lambda} + \frac{1}{R})) + \frac{|y-x|^2}{\lambda T}.$$

- $$|b(t,\omega,X_t)|^2 + \sum_{j=1}^{\infty} \left|\sigma_j(,\omega,X_t)\right|^2 \leq c\,|X_t|^2$$

- $$\sum_{j=1}^{\infty} E_s(\left|\sigma_j(s,\omega,X_s) - \sigma_j(t,\omega,X_t)\right|^2) \leq L^2(t-s)E_t(\sup_{s\leq u\leq t} |X_u|^2)$$

- For some $0 < \lambda \leq 1 \leq \gamma$

$$\lambda \left[\frac{\max_{i=1,n}\left|X_t^i\right|}{\min_{i=1,n}\left|X_t^i\right|}\right]^2 \leq \sigma\sigma^*(t,\omega,X_t) \leq \gamma\,|X_t|^2$$

Theorem

In the Log-normal case we have

$$P(|X_t - x_t| \leq Rm(x_t),\ \textit{for all } t \leq T)$$

$$\geq \exp\left\{-C_T(\theta(x)+\theta(y))^2\left(1+\frac{d^2(x,y)}{T\lambda}\right)\right\}$$

where

$$x_i(t) = x_i^{1-t/T} y_i^{t/T}, \quad m(x_t) = \min_{i=1,\dots,n} |x_i(t)|,$$

$$d(x,y) = \max_{i=1,n}\left|\ln y^i - \ln x^i\right|, \qquad \theta(x) = \max_{i,j} \frac{x^i}{x^j}.$$

- The bound in the Elliptic case

$$\exp\left\{-C_T\left(K+\frac{|y-x|^2}{\lambda T}\right)\right\}.$$

- The bound in the Log-normal case

$$\exp\left\{-C_T(\theta(x)+\theta(y))^2\left(1+\frac{d^2(x,y)}{\lambda T}\right)\right\}$$

$$d(x,y)=\max_{i=1,n}\left|\ln y^i-\ln x^i\right|,\qquad \theta(x)=\max_{i,j}\frac{x^i}{x^j}.$$

The *good* distance is the Logarithmic one

$$d(x, y) = \max_{i=1,n} \left| \ln y^i - \ln x^i \right|, \qquad \theta(x) = \max_{i,j} \frac{x^i}{x^j}.$$

Corollary

In the Log-normal case, we obtain:

$$P(\sup_{t \leq T} d(X_t, x_t) \leq R) \geq \exp(-C_T(\theta(x) + \theta(y))^2(1 + \frac{d^2(x, y)}{T\lambda})).$$

- Let $y \in R^n$, $\varepsilon > 0$ and $\mu \geq 1$. We define $L^d_y(\mu,\varepsilon)$ to be the class of positive functions such that

$$f(y) \leq \mu f(x) \quad \forall x \in B^d_\varepsilon(y).$$

Theorem

For every $f \in L^d_{y_0}(\mu,\varepsilon)$ one has

$$E(f(X^x_T)1_{B_\varepsilon(y_0)}(X^x_T)\ ) \geq \frac{1}{\mu^2}\int_{B_\varepsilon(y_0)} f(z)\exp\left\{-\frac{C'_T K(x,z)}{d(x,y_0)^2\varepsilon^2}\left(K+\frac{d(x,z)^2}{\lambda T}\right)\right\})dz.$$

where $d(x,y)$ is a good distance.

Theorem

Estimations of the Distribution Functions

$$P[X_T^i > y_0^i,\ i = 1, ..., n] \\ \geq \frac{1}{\mu^2} \int_{\{z^i > y_0^i, i=1,...,n\}} \exp\left\{ -\frac{C_T' K(x,z)}{d(x,y_0)^2 \varepsilon^2} \left(K + \frac{d(x,z)^2}{\lambda T} \right) \right\} dz.$$

where $d(x, y)$ is a good distance.

- Log-normal diffusion

$$\frac{dX_t^i}{X_t^i} = \sum_{j=1} \sigma_j^i(t,\omega) dW_t^j + b^i(t,\omega)dt, \quad i = 1, n. \tag{2}$$

- $$Y_t = \sum_{j=1}^{n} X_t^j.$$

- $$Z_t = \int_0^t \sum_{j=1}^{n} X_s^j ds.$$

Theorem

$$E[(Y_T - K)_+] \geq \int_0^{\infty} (z-K)_+ exp\left\{ -C_t\,(1+K(x,z))\left(1+\frac{1}{(z-K)_+}\right)\right\} dz,$$

where

$$K(x,z) = \left| ln \frac{x_1 + \cdots + x_n}{z} \right|^2$$

Theorem

$$E[(Z_T - K)_+] \geq \int_0^{\infty} (z-K)_+ exp\left\{ -C_t \left(1 + K'(x,z)\right) \left(1 + \frac{1}{(z-K)_+}\right) \right\} dz,$$

where $K'(x, z)$ is rather complicated!!!!!!!

Thanks!!!!!

Merci!!!!!!

¡¡¡¡Thanks!!!!!!

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